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Dequantization of Mathematics, idempotent semirings and fuzzy sets

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1. Introduction. The traditional mathematics over numerical fields can be dequantized as the Planck constant $\hbar$ tends to zero taking pure imaginary values. This dequantization leads to the so-called Idempotent Mathematics based on replacing the usual arithmetic operations by a new set of basic operations (e.g., such as maximum or minimum), that is on the concepts of idempotent semifield and semiring. Typical examples are given by the so-called $(\max, +)$ algebra $\mathbf{R}_{\max}$ and $(\min, +)$ algebra $\mathbf{R}_{\min}$. Let $\mathbf{R}$ be the field of real numbers. Then $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. Similarly $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min, \odot = +$. The new addition $\oplus$ is idempotent, i.e., $x \oplus x = x$ for all elements x. Some problems that are nonlinear in the traditional sense turn out to be linear over a suitable idempotent semiring (idempotent superposition principle [1]). For example, the Hamilton-Jacobi equation (which is an idempotent version of the Schrödinger equation) is linear over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$.

The basic paradigm is expressed in terms of an *idempotent correspondence principle* [2].This principle is similar to the well-known correspondence principle of N. Bohr in quantum theory (and closely related to it). Actually, there exists a heuristic correspondence between important, interesting and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition). For example, the well-known Legendre transform can be treated as an $\mathbf{R}_{\max}$-version of the traditional Fourier transform (this observation is due to V. P. Maslov).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over rings, its "shadow", the Idempotent Mathematics, appears. This "shadow" stands approximately in the same relation to the traditional mathematics as classical physics to quantum theory. In many respects Idempotent Mathematics is simpler than the traditional one. However, the transition from traditional concepts and results to their idempotent analogs is often nontrivial.

In this talk a brief survey of basic ideas of Idempotent Mathematics is presented. Relations between this theory and the theory of fuzzy sets as well as the possibility theory and some applications (including computer applications) are discussed. Hystorical surveys and the corresponding references can be found in [2]–[5].

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2. Semirings, semifields, and idempotent dequantization. Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;

- the addition $\oplus$ is commutative;

- the multiplication $\odot$ is distributive with respect to the addition $\oplus$: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in S$.

The semiring is *commutative* if the multiplication $\odot$ is commutative. A *unity* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral elements $\mathbf{0}$ and $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible.

The following examples are important. Let $\mathbf{P}$ be the segment $[0, 1]$ equipped with the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$; then $\mathbf{P}$ is a commutative idempotent semiring (but not a semifield). The subset $\mathbf{B} = \{0, 1\}$ in M equipped with the same operations is the well-known Boolean algebra which is an idempotent semifield. In this case $\oplus$ and $\odot$ are the usual Boolean operations (disjunction and conjunction). In the general case the semiring addition and multiplication could be treated as generalized logical (Boolean) operations.

Let $\mathbf{R}$ be the field of real numbers and $\mathbf{R}_+$ the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). The change of variables $x \mapsto u = h \ln x$, $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to S = \mathbf{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbf{R}$ to S by Φ_h, i.e., let $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$, $\mathbf{0} = -\infty = \Phi_h(0)$, $\mathbf{1} = 0 = \Phi_h(1)$. It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$ and S forms a semiring with respect to addition $u \oplus v = \max\{u, v\}$ and multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by $\mathbf{R}_{\max}$; it is idempotent. The semiring $\mathbf{R}_{\max}$ is actually a commutative semifield. This construction is due to V.P. Maslov [1]; now it is known as *Maslov's dequantization*.

The analogy with quantization is obvious; the parameter h plays the rôle of the Planck constant, so $\mathbf{R}_+$ (or $\mathbf{R}$) can be viewed as a "quantum object" and $\mathbf{R}_{\max}$ as the result of its "dequantization". A similar procedure gives the semiring $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$; in this case $\mathbf{0} = +\infty$, $\mathbf{1} = 0$. The semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$ are isomorphic. Connections with physics and imaginary values of the Planck constant are discussed in [4]. The commutative idempotent semiring $\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$, $\odot = \min$ can be obtained as a result of a "second dequantization" of $\mathbf{R}$ (or $\mathbf{R}_+$). Dozens of interesting examples of nonisomorphic idempotent semirings may be cited as well as a number of standard methods of deriving new semirings from these (see, e.g., [2]–[5]).

Idempotent dequantization is a generalization of Maslov's dequantization. This is a passage from fields to idempotent semifields and semirings in mathematical constructions and results. The idempotent correspondence principle (see Introduction and [2, 4]) often works for this idempotent dequantization.

3. Idempotent Analysis. Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, zero $\mathbf{0}$, and unit $\mathbf{1}$. The set S is supplied with the *standard partial order* $\preccurlyeq$: by definition, $a \preccurlyeq b$ if and only if $a \oplus b = b$. Thus all elements of S are positive: $\mathbf{0} \preccurlyeq a$ for all $a \in S$. Due to the existence of this order, Idempotent Analysis is closely related to the lattice theory, the theory of vector lattices, and the theory of ordered spaces. Moreover, this partial order allows to simulate a number of basic notions and results of Idempotent Analysis at the purely algebraic level.

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $\mathcal{B}(X, S)$ of functions $X \to S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\, dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where $\varphi \in \mathcal{B}(X, S)$. Indeed, a Riemann sum of the form $\sum_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus_i \varphi(x_i) \odot \sigma_i = \max_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument. Formula (1) defines the idempotent integral not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An idempotent measure on X is defined by $m_{\psi}(Y) = \sup_{x \in Y} \psi(x)$, where $\psi \in \mathcal{B}(X, S)$. The integral with respect to this measure is defined by

$$I_{\psi}(\varphi) = \int_X^{\oplus} \varphi(x)\, dm_{\psi} = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)). \tag{2}$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order $\preccurlyeq$ is opposite to the conventional order $\leqslant$, so in this case equation (2) assumes the form

$$\int_X^{\oplus} \varphi(x)\, dm_{\psi} = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)), \tag{3}$$

where inf is understood in the sense of the conventional order $\leqslant$.

The functionals $I(\varphi)$ and $I_{\psi}(\varphi)$ are linear over S; their values correspond to limits of Lebesgue (or Riemann) sums. The formula for $I_{\psi}(\varphi)$ defines the idempotent scalar product of the functions ψ and φ. Various idempotent functional spaces and an idempotent version of the theory of distributions can be constructed on the basis of the idempotent integration, see, e.g., [1], [3]–[5]. The analogy of idempotent and probability measures leads to spectacular parallels between optimization theory and probability theory. For example, the Chapman–Kolmogorov equation corresponds

to the Bellman equation (see, e.g., [6, 5]). Many other idempotent analogs may be cited (in particular, for basic constructions and theorems of functional analysis [4]).

4. The superposition principle and linear problems. Basic equations of quantum theory are linear (the superposition principle). The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However it is linear over the semirings $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$. Also, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings (V. P. Maslov's idempotent superposition principle), see [1, 3]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F [7]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. In [7], it was shown that main optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc.

Linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ (and $\mathbf{R}_{\max}$) is closely related to the (conventional) linearity of the Schrödinger equation, see [4] for details.

5. Correspondence principle for algorithms and their computer implementations. The idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [2]. In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. It is well-known that algorithms of linear algebra are convenient for parallel computations; so their idempotent analogs accept a parallelization. This is a regular way to use parallel computations for many problems including basic optimization problems. It is convenient to use universal algorithms which do not depend on a concrete semiring and its concrete computer model. Software implementations for universal semiring algorithms are based on object-oriented and generic programming; program modules can deal with abstract (and variable) operations and data types, see [2, 8] for details.

The most important and standard algorithms have many hardware realizations in the form of technical devices or special processors. These devices often can be used as prototypes for new hardware units generated by substitution of the usual arithmetic operations for its semiring analogs, see [2] for details. Good and efficient technical ideas and decisions can be transposed from prototypes into new hardware units. Thus the correspondence principle generates a regular heuristic method for hardware design. Note that to get a patent it is necessary to present the so-called "invention formula", that is to indicate a prototype for the suggested device and the difference between these devices.

6. Idempotent interval analysis. An idempotent version of the traditional interval analysis is presented in [9]. Let S be an idempotent semiring equipped with the standard partial order (see the beginning of Section 3). A *closed interval* in S is a subset of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in S | \underline{\mathbf{x}} \preccurlyeq x \preccurlyeq \overline{\mathbf{x}}\}$, where the elements $\underline{\mathbf{x}} \preccurlyeq \overline{\mathbf{x}}$

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are called *lower* and *upper bounds* of the interval $\mathbf{x}$. A *weak interval extension $I(S)$* of the semiring S is the set of all closed intervals in S endowed with operations $\oplus$ and $\odot$ defined as $\mathbf{x} \oplus \mathbf{y} = [\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}, \overline{\mathbf{x}} \oplus \overline{\mathbf{y}}]$, $\mathbf{x} \odot \mathbf{y} = [\underline{\mathbf{x}} \odot \underline{\mathbf{y}}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}]$; the set $I(S)$ is a new idempotent semiring with respect to these operations. It is proved that basic problems of idempotent linear algebra are polynomial, whereas in the traditional interval analysis problems of this kind are generally NP-hard. Exact interval solutions for the discrete stationary Bellman equation (this is the matrix equation discussed in Section 4) and for the corresponding optimization problems are constructed and examined.

7. Generalized fuzzy sets. Let Ω be the so-called universe consisting of "elementary events" and S an idempotent semiring. Denote by $\mathcal{F}(S)$ the set of functions defined on Ω and taking their values in S; then $\mathcal{F}(S)$ is an idempotent semiring with respect to the pointwise addition and multiplication of functions. We shall say that elements of $\mathcal{F}(S)$ are *generalized fuzzy sets*. See also [13]. We have the well known classical definition of fuzzy sets (L.A. Zadeh [10]) if $S = \mathbf{P}$, where $\mathbf{P}$ is the segment $[0, 1]$ with the semiring operations $\oplus = \max$ and $\odot = \min$, see Section 2. Of course, functions from $\mathcal{F}(\mathbf{P})$ taking their values in the Boolean algebra $\mathbf{B} = \{0, 1\} \subset \mathbf{P}$ correspond to traditional sets from Ω and semiring operations correspond to standard operations for sets. In the general case if S has neutral elements $\mathbf{0}$ and $\mathbf{1}$ (and $\mathbf{0} \neq \mathbf{1}$), then functions from $\mathcal{F}(S)$ taking their values in $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\} \subset \mathbf{S}$ can be treated as traditional subsets in Ω. If S is a lattice (i.e. $x \odot y = \inf\{x, y\}$ and $x \oplus y = \sup\{x, y\}$), then generalized fuzzy sets coincide with L-fuzzy sets in the sense of J.A. Goguen [11]. The set $I(S)$ of intervals is an idempotent semiring (see Section 6), so elements of $\mathcal{F}(I(S))$ can be treated as interval (generalized) fuzzy sets.

It is well known that the classical theory of fuzzy sets is a basis for the theory of possibility [12]. Of course, it is possible to develop a similar generalized theory of possibility starting from generalized fuzzy sets. In general the generalized theories are noncommutative; they seem to be more qualitative and less quantitative with respect to the classical theories presented in [10, 12]. We see that Idempotent Analysis and the theory of (generalized) fuzzy sets and possibility have the same objects, i.e. functions taking their values in semirings. However, basic problems and methods could be different for these theories (like for the measure theory and the probability theory).

References

[1] V. P. Maslov, *Méthodes opératorielles*, Mir, Moscou, 1987.

[2] G.L. Litvinov and V.P. Maslov, *Correspondence principle for idempotent calculus and some computer applications*, (IHES/M/95/33), Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette 1995, also in: [5], 420–443. E-print math.GM/0101021 (http://arXiv.org).

[3] V.N. Kolokoltsov and V. P. Maslov, *Idempotent Analysis and Applications*, Kluwer Acad. Publ., Dordrecht, 1997.

[4] G. L. Litvinov, V. P. Maslov, G. B. Shpiz, *Idempotent functional analysis: an algebraic approach*, Mathematical Notes, **69**, no. 5 (2001), 696–729, E-print math.FA/0009128 (http://arXiv.org).

[5] J. Gunawardena (ed.), *Idempotency*, Publ. of the Newton Institute, Cambridge Univ. Press, Cambridge, 1998.

[6] P. Del Moral, *A survey of Maslov optimization theory*, In [3].

[7] B. A. Carré, *An algebra for network routing problems*, J. Inst. Math. Appl., **7** (1971), 273–294.

[8] G.L. Litvinov and E.V. Maslova, *Universal numerical algorithms and their software implementation*, Programming and Computer Software, **26**, no. 5 (2000), 275–280. E-print math.SC/0102114 (http://arXiv.org).

[9] G.L. Litvinov and A.N. Sobolevskii, *Idempotent interval analysis and optimization problems*, Reliable Computing, **7**, no. 5 (2001), 353–377. E-print math.SC/0101080 (http://arXiv.org).

[10] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.

[11] J.A. Goguen, *L-fuzzy sets*, J. of Math. Anal. Appl., **18**, no. 1 (1967), 145–174.

[12] L.A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, **1** (1978), 3–28.

[13] J.S. Golan, *Semirings and their applications*, Kluwer Acad. Publ., Dordrecht, 1999.